MINIMAL POLYNOMIAL IDENTITIES FOR RIGHT-SYMMETRIC ALGEBRAS

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Abstract. An algebra A with multiplication $A\times A \to A, (a,b)\mapsto a\circ b,$ is called right-symmetric, if $a\circ(b\circ c)-(a\circ b)\circ c = a\circ(c\circ b)-(a\circ c)\circ b,$ for any $a,b,c\in A.$ The multiplication of right-symmetric Witt algebras $W_n = \{u\partial_i : u \in U, U = \mathcal{K}[x_1^{\pm 1},\ldots,x_n^{\pm 1}],$ or $= \mathcal{K}[x_1,\ldots,x_n], i=1,\ldots,n\},$ $p=0,$ or $W_n(\mathbf{m}) = \{u\partial_i : u\in U, U=O_n(\mathbf{m})\},$ $p>0,$ are given by $u\partial_i\circ v\partial_j = v\partial_j(u)\partial_i.$ An analogue of the Amitsur-Levitzki theorem for right-symmetric Witt algebras is established. Right-symmetric Witt algebras of rank n satisfy the standard right-symmetric identity of degree $2n+1:$ $\sum_{\sigma\in Sym_{2n}} sign(\sigma) a_{\sigma(1)}\circ(a_{\sigma(2)}\circ\cdots(a_{\sigma(2n)}\circ a_{2n+1})\cdots)=0.$ The minimal degree for left polynomial identities of $W_n^{rsym}, W_n^{+rsym}, p=0,$ is $2n+1.$ The minimal degree of multilinear left polynomial identity of $W_n(\mathbf{m}), p>0,$ is also $2n+1.$ All left polynomial (also multilinear, if $p>0$) identities of right-symmetric Witt algebras of minimal degree are linear combinations of left polynomials obtained from standard ones by permutations of arguments.

1. Introduction

According to the Amitsur-Levitzki theorem [1] the matrix algebra Mat_n satisfies a standard polynomial identity of degree $2n:$

$$\sum_{\sigma\in Sym_{2n}} sign\,\sigma\, a_{\sigma(1)}\circ\cdots a_{\sigma(2n)} = 0,$$

where $a\circ b$ is a usual matrix multiplication. Moreover, Mat_n has no polynomial identity of degree less that $2n.$ For details on polynomial identities of associative algebras see for example, [12].

An algebra $W_1=\{e_i : e_i\circ e_j = (i+1)e_{i+j}, i,j\in \mathbf{Z}\}$ is right-symmetric. Since its Lie algebra is isomorphic to a Witt algebra $W_1 = \{e_i : [e_i,e_j]=(j-i)e_{i+j}\}$ we call it as right-symmetric Witt algebra of rank 1 and denote by $W_1^{rsym}.$ This algebra satisfies a right-symmetric identity

$$a\circ(b\circ c)-(a\circ b)\circ c = a\circ(c\circ b)-(a\circ c)\circ b$$

and a left-commutativity identity

$$a\circ(b\circ c) = b\circ(a\circ c). \tag{1}$$

Such algebras are called *Novikov* [9], [10], [11].

There is a generalisation of the Witt algebra to the many valuables case. Let U be an associative commutative algebra with a set of commuting derivations $\mathcal{D}=\{\partial_i : i=1,\dots,n\}$. For any $u\in U$, an endomorphism $u\partial_i : U\to U$, such that $(u\partial_i)(v)=u\partial_i(v)$, is a derivation of U. Denote by $U\mathcal{D}$ a space of derivations $\sum_{i=1}^n u_i\partial_i$. Endow this space by multiplication

$$u\partial_i\circ v\partial_j = v\partial_j(u)\partial_i.$$

We obtain a right-symmetric algebra $U\mathcal{D}$. This algebra is called a *a right-symmetric Witt algebra generated on U and $\mathcal{D}$.*

In our paper, U is $\mathcal{K}[x_1,\dots,x_n]$, or Laurent polynomial algebra $\mathcal{K}[x_1,^{\pm1},\dots,x_n^{\pm1}]$, or a divided power algebra $O_n(\mathbf{m})=\{x^\alpha : x^\alpha x^\beta = \binom{\alpha+\beta}{\alpha}x^{\alpha+\beta}\}$, if the characteristic of $\mathcal{K}$ is $p>0$. As a Lie algebra the Witt algebra of the rank n is defined as a Lie algebra of derivations of U. The multiplication $u\partial_i\circ v\partial_j = v\partial_j(u)\partial_i$ satisfies the right-symmetry identity. Obtained right-symmetric Witt algebras of rank n, are denoted by W_n^{rsym} or W_n^{+rsym} or $W_n(\mathbf{m})^{rsym}$, depending on $U=\mathcal{K}[x_1^{\pm1},\dots,x_n^{\pm1}]$, or $\mathcal{K}[x_1,\dots,x_n]$, or $O_n(\mathbf{m})$. It is easy to notice that the right-symmetric Witt algebras of rank n do *not* satisfy the left-commutativity identity if $n>1$.

We are interested in the analogues of the left-symmetric identities for the case of many valuables. We suggest two ways to solve this problem.

In the first way we endow a vector space of the Witt algebra with two multiplications: the multiplication $(a,b)\mapsto a\circ b$, mentioned above and the second multiplication defined by $u\partial_i * v\partial_j = \partial_i(u)v\partial_j$. We obtain an algebra with the following identities

$$a\circ(b\circ c)-(a\circ b)\circ c - a\circ(c\circ b)+(a\circ c)\circ b = 0,$$

$$a*(b*c)-b*(a*c)=0, \tag{2}$$

$$a\circ(b*c)-b*(a\circ c)=0,$$

$$(a*b-b*a-a\circ b+b\circ a)*c=0,$$

$$(a\circ b-b\circ a)*c+a*(c\circ b)-(a*c)\circ b-b*(c\circ a)+(b*c)\circ a=0.$$

Notice that (2), for the multiplication $*$, is similar to the identity (1) for the multiplication $\circ$.

In the second way we save right-symmetric multiplication $(a,b)\mapsto a\circ b$, and try to construct an identity for its left multiplication operators. For $a\in A$, denote by r_a and l_a operators of right and left multiplications on $A:$ $br_a=b\circ a$, $bl_a=a\circ b$. In terms of right and

left multiplication operators the right-symmetry identity is equivalent to the following conditions

$$[r_a, r_b] = r_{[a,b]}, \quad [r_a, l_b] = l_a l_b - l_{b \circ a}, \ \forall a, b \in A.$$

Our main result is the following. For right-symmetric Witt algebras of any rank n for left multiplication operators the following *standard polynomial identity of degree* $2n$ holds:

$$\sum_{\sigma \in Sym_{2n}} sign\, \sigma\, l_{a_{\sigma(1)}} \cdots l_{a_{\sigma(2n)}} = 0,$$

or in terms of multiplication $\circ$, the following *left polynomial identity of degree* $2n+1$ is valid:

$$\sum_{\sigma \in Sym_{2n}} sign\, \sigma\, a_{\sigma(2n)} \circ (a_{\sigma(2n-1)} \circ \cdots \circ (a_{\sigma(2)} \circ (a_{\sigma(1)} \circ a_0)) \cdots) = 0.$$

We prove that in the space of left polynomial identities the degree $2n+1$ for right-symmetric Witt algebras of rank n is *minimal* (in the case of $p > 0$ we suppose also that polynomials are multilinear). We also prove that the left polynomial identities of right-symmetric Witt algebras of rank n of minimal degree *can be obtained from standard polynomials by permutation and linear combination operations.*

Notice that, for $n = 1$ the identity $s_2^{rsym}(a_0, a_1, a_2) = 0$, coincides with the Novikov identity

$$a_0 \circ (a_1 \circ a_2) - a_1 \circ (a_0 \circ a_2) = 0.$$

So, we can consider right-symmetric algebras that satisfy standard right-symmetric identity

$$\sum_{\sigma \in Sym2n} sign\, \sigma\, a_{\sigma(1)} \circ a_{\sigma(2)} \circ \cdots a_{\sigma(2n)} \circ a_0 = 0, \tag{3}$$

as a generalisation of Novikov algebras. This class of algebras includes Witt algebras in the many valuables case.

In our proof we use some properties of Laurent or divided power polynomials. The identity (1) is true for any associative commutative algebra U and for a set $\mathcal{D}$ with one derivation ∂_1. We believe that the identity (3) holds for any associative commutative algebra U with a set of commuting derivations $\mathcal{D}$ with n derivations $\partial_i, i = 1, \ldots, n$.

Conjecture. *Any simple right-symmetric algebra over an algebraically closed field of characteristic* $p = 0$ *or* $p > 2n+1$ *with minimal left polynomial identity* s_{2n}^{rsym} *is isomorphic to one of the following algebras:*

- *a Witt algebra generated by some associative commutative algebra* U *with a set of linear independent commuting derivations* $\mathcal{D}$ *with* n *derivations,*

- *or their deformations.*

About deformations of right-symmetric algebras and the description of local deformations of $A = W_n^{rsym}, W_n^{+rsym}$, if $p = 0$, or $W_n(\mathbf{m})$, if $p > 0$, see [6], [7]. As an example let us give some right-symmetric deformations of $A = W_1 = \{a = u\partial : u \in \mathcal{K}[x^{\pm 1}]\}$.

The space of local deformations of A is 4-dimensional and generated by classes of the following right-symmetric 2-cocycles

$$\psi^1(u\partial, v\partial) = x^{-1}uv\partial,$$
$$\psi^2(u\partial, v\partial) = x^{-1}\partial(u)v\partial,$$
$$\psi^3(u\partial, v\partial) = (u - x\partial(u))\partial(v)\partial,$$
$$\psi^4(u\partial, v\partial) = \partial(u)\partial(v)\partial.$$

If $\Psi_1 = \sum_{k=1}^4 \varepsilon_k\psi^k$ is a 4-parametrical local deformation of A, is it possible to construct prolongations $\Psi_l = \sum_{|\mathbf{i}|=l} \epsilon^{\mathbf{i}}\psi^{\mathbf{i}}$, where $\mathbf{i} = (i_1, i_2, i_3, i_4), |\mathbf{i}| = i_1 + i_2 + i_3 + i_4$? In other words, is it possible to find $\psi^{\mathbf{i}} \in C^2_{rsym}(A, A)$, such that a new multiplication

$$a \circ_\varepsilon b = a \circ b + \sum_l \Psi_l,$$

will be a right-symmetric multiplication over a field $\mathcal{K}((\varepsilon))$? The answer is: Ψ_1 can be prolongated to a global deformation if and only if $\varepsilon_1\varepsilon_4 + \varepsilon_2\varepsilon_3 = 0$. We give prolongation formulas for some special cases.

The local deformation $\varepsilon_1\psi^1 + \varepsilon_2\psi^2$ of W_1 has a trivial prolongation:

$$(u\partial, v\partial) \mapsto \partial(u)v + \varepsilon_1 x^{-1}uv + \varepsilon_2 x^{-1}\partial(u)v,$$

is a right-symmetric multiplication. This algebra was obtained by Osborn [11]. His results confirm our conjecture for the case of $n = 1$. Notice that, cocycles ψ^3, ψ^4 do not satisfy left-commutativity identity. They are not Novikov cocycles [2]. Each of these cocycles have the following prolongations:

$$(a, b) \mapsto \partial(a)b + [x\partial, a](\sum_i \varepsilon_3^i(-1)^i x^i\partial^i/\{(\varepsilon_3 + 1)\cdots(i\varepsilon_3 + 1)\})(b), \tag{4}$$

$$(a, b) \mapsto \partial(a)\sum_i \varepsilon_4^i\partial^i(b).$$

In the case of (4) we should change expressions like $(i\varepsilon_4 + 1)^{-1}$ to formal series $1 - i\varepsilon_3 + i^2\varepsilon_3^2 - i^3\varepsilon_3^3 + \cdots$. Then we obtain a formal power serie

$$\begin{gathered}(a, b) \mapsto \partial(a)b \\ -\varepsilon_3[x\partial, a]\partial(b) \\ +\varepsilon_3^2[x\partial, a](x\partial(b) + x^2\partial^2)(b) \\ -\varepsilon_3^3[x\partial, a](x\partial + 3x^2\partial^2 + x^3\partial^3)(b)\end{gathered}$$

$$+\varepsilon_3^4[x\partial,a](x\partial+7x^2\partial^2+6x^3\partial^3+x^4\partial^4)(b)$$
$$-\varepsilon_3^5[x\partial,a](x\partial+63x^2\partial+25x^3\partial^3+10x^4\partial^4+x^5\partial^5)(b)+\cdots.$$

This is one of the prolongations of the local deformation $-\varepsilon_3[x\partial,a]\partial(b)$. It will be interesting to construct prolongation formulas for a linear combination of cocycles and find polynomial identities of obtained algebras. It is also interesting to find right polynomial identities of right-symmetric algebras. Right multiplication operators satisfy Lie algebraic conditions and in this case one can expect Lie algebraic difficulties (see [13], [14]). Let us mention that an identity of degree 5 for Lie algebra W_1^+ is true also for right multiplication operators:

$$\sum_{\sigma\in Sym_4} sign\,\sigma\, r_{a_{\sigma(1)}}r_{a_{\sigma(2)}}r_{a_{\sigma(3)}}r_{a_{\sigma(4)}}=0.$$

Moreover, for right-symmetric Witt algebra $U\mathcal{D}$, $\mathcal{D}=\{\partial\}$, the following right polynomial identity of degree 3 takes place

$$\sum_{\sigma\in Sym_3} sign\,\sigma(a_{\sigma(1)}\circ a_{\sigma(2)})\circ a_{\sigma(3)}=0.$$

For a right-symmetric algebra $U\mathcal{D}$, where $\mathcal{D}=\{\partial_1,\partial_2\}$, the following right polynomial identity of degree 7 is true

$$\sum_{\sigma\in Sym_7} sign\,\sigma(((((a_{\sigma(1)}\circ a_{\sigma(2)})\circ a_{\sigma(3)})\circ a_{\sigma(4)})\circ a_{\sigma(5)})\circ a_{\sigma(6)})\circ a_{\sigma(7)}=0.$$

2. Right-symmetric algebras

Let A be an algebra with multiplication $A\times A\to A, (a,b)\mapsto a\circ b$. Let $(a,b,c)=a\circ(b\circ c)-(a\circ b)\circ c$ be an associator of elements $a,b,c\in A$. Associative algebras are defined by condition $(a,b,c)=0$, for any $a,b,c\in A$. Right-symmetric algebras are defined by identity

$$(a,b,c)=(a,c,b),$$

i.e., by identity

$$a\circ(b\circ c)-(a\circ b)\circ c=a\circ(c\circ b)-(a\circ c)\circ b.$$

The left-symmetric identity is

$$(a,b,c)=(b,a,c).$$

There is one-to-one correspondence between right-symmetric and left-symmetric algebras. Namely, if $(a,b)\to a\circ b$, is right(left)-symmetric, then a new multiplication $(a,b)\mapsto b\circ a$, is left(right)-symmetric. In our paper left-symmetric algebras are not considered. Right-symmetric algebras are called sometimes as Vinberg-Kozsul algebras [15], [8]. Right-symmetric algebra A is Lie-admissible, i.e., under commutator $[a,b]=a\circ b-b\circ a$, we obtain a Lie algebra.

Any associative algebra is right-symmetric. In a such cases, we will use notations like A^{ass}, if we consider A as an associative algebra and A^{rsym} if we consider A as right-symmetric algebra. Similarly, for a right-symmetric algebra A the notation A^{rsym} will mean that we use only right-symmetric structure on A and A^{lie} stands for a Lie algebra structure under commutator $(a,b)\mapsto [a,b]$.

In terms of operators of right multiplication r_a and left multiplication l_a of the algebra A,

$$ar_b = a\circ b,\ al_b = b\circ a,$$

right-symmetry identities are equivalent to the following conditions:

$$[r_a,r_b]-r_{[a,b]}=0,$$

$$[r_a,l_b]-l_al_b+l_{b\circ a}=0.$$

Let $A_r=\{a_r : a\in A\}$ and $A_l=\{a_l : a\in A\}$ be two copies of A, and $\mathcal{A}=A_r\oplus A_l$ be their direct sum. Let $T(\mathcal{A})=\mathcal{K}\oplus\mathcal{A}\oplus\mathcal{A}\otimes\mathcal{A}\oplus\cdots$ be the tensor algebra of $\mathcal{A}$. A universal enveloping algebra of A, denoted by $U(A)$, is defined as a factor-algebra of $T(A_r\oplus A_l)$ over an ideal generated by $[a_r,b_r]-[a,b]_r, [a_r,b_l]-a_lb_l+(b\circ a)_l$. Denote elements of $U(A)$ corresponding to a_r, a_l by r_a, l_a.

Let B be a subalgebra of A. Let

$$Z_A^{l.ass}(B)=\{a\in A : (a,b_1,b_2)=0,\ \forall b_1,b_2\in B\},$$

$$Z_A^{r.ass}(B)=\{a\in A : (b_1,b_2,a)=0,\ \forall b_1,b_2\in B\},$$

be *left and right associative centralisers of* B *in* A. Let

$$N_A^{l.ass}(B)=\{a\in A : (a,b_1,b_2)\in B, \forall b_1,b_2\in B\},$$

$$N_A^{r.ass}(B)=\{a\in A : (b_1,b_2,a)\in B, \forall b_1,b_2\in B\},$$

be *left and right normalisers of* B *in* A. It is clear that

$$Z_A^{l.ass}(B)\subseteq N_A^{l.ass}(B),$$

$$Z_A^{r.ass}(B)\subseteq N_A^{r.ass}(B).$$

Let

$$Z_A^{left}(B)=\{a\in A : a\circ b=0,\ \forall b\in B\},$$

$$Z_A^{right}(B)=\{a\in A : b\circ a=0,\ \forall b\in B\},$$

be *left and right centralisers of* B *in* A and

$$N_A^{left}(B)=\{a\in A : a\circ b\in B,\ \forall b\in B\},$$

$$N_A^{right}(B)=\{a\in A : b\circ a\in B,\ \forall b\in B\},$$

be *left and right normalisers of* B *in* A. We have

$$Z_A^{left}(B)\subseteq Z_A^{l.ass}(B),$$

$$Z_A^{right}(B) \subseteq Z_A^{r.ass}(B),$$
$$N_A^{left}(B) \subseteq N_A^{l.ass}(B),$$
$$N_A^{right}(B) \subseteq N_A^{r.ass}(B).$$

For the left cases and if $A = B$ we reduce these denotions: $Z(A) = Z_A^{left}(A)$, $N(B) = N_A^{left}(B)$, $Z^{right}(A) = Z_A^{right}(A)$, $N^{right}(A) = N_A^{right}(A)$, $Z^{l.ass}(A) = Z_A^{l.ass}(A), N^{l.ass}(A) = N_A^{l.ass}(A)$, $Z^{r.ass}(A) = Z_A^{r.ass}(A), N^{r.ass}(A) = N_A^{r.ass}(A)$.

We call $Z(A)$ and $Z^{right}(A)$ as left and right centers of A. We call also $Z^{l.ass}(A)$ and $Z^{r.sym}(A)$ as left and right associative centers of A.

Left (right) associative centers are close under multiplication $\circ$. To see this let us consider for simplicity the case of left associative centers. Suppose that $X, Y \in Z^{l.ass}(A)$. Then according to the right-symmetric identity

$$(X \circ Y) \circ (a \circ b) =$$
$$X \circ (Y \circ (a \circ b)) = X \circ ((Y \circ a) \circ b) = (X \circ (Y \circ a)) \circ b =$$
$$((X \circ Y) \circ a) \circ b.$$

So, $X \circ Y \in Z^{l.ass}(A)$, and $Z^{left}(A)$ is a subalgebra of A.

Notice that $Z(A)$ and $N(Z(A))$ are also subalgebras of $A:$

$$(z_1 \circ z_2) \circ a = z_1 \circ (z_2 \circ a) - (z_1, a, z_2) = 0,$$

$$((n_1 \circ n_2) \circ z) \circ a = (n_1 \circ (n_2 \circ z)) \circ a - (n_1, z, n_2) \circ a = 0,$$

for any $a \in A, z_1, z_2 \in Z(A), n_1, n_2 \in N(Z(A))$. The same is true for $Z^{right}(A)$ and $N^{right}(Z^{right}(A))$.

Proposition 2.1. *If $z \in Z(A)$, then r_z is a derivation of A.*

Proof. Since $z \circ b = 0$, we have $a \circ (z \circ b) = 0$. According right-symmetric identity

$$(a \circ b) \circ z = a \circ (b \circ z) + (a \circ z) \circ b,$$

for any $a, b \in A$.•

Proposition 2.2. *For any $z \in Z(A), a \in N(Z(A))$, and for any $b \in A$,*

$$a \circ (b \circ z) = (a \circ b) \circ z.$$

Proof. Let $z \in Z(A)$. Then $z \circ b = 0$, and $a \circ (z \circ b) = 0$. Let $a \in N(Z(A))$. Then $(a \circ z) \circ b = 0$. So,

$$a \circ (b \circ z) - (a \circ b) \circ z = a \circ (z \circ b) - (a \circ z) \circ b = 0.•$$

Corollary 2.3. *For* $N = N(Z(A))$,

$$Z^{left}(A) \subseteq Z^{r.ass}(N).$$

Proof. Evident.

Corollary 2.4. *For any* $a_1, \dots, a_{n-1} \in N(Z(A))$, *and* $a_n \in A$, $z \in Z(A)$, *the following relation takes place*

$$a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n \circ z = (a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n) \circ z.$$

Proof. For $n = 2$, the statement follows from the lemma. Suppose that this is also true for $n-1$. Then by our lemma

$$a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ (a_n \circ z)) \cdots) =$$

$$a_1 \circ \{(a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots) \circ z\} =$$

$$\{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ z. \bullet$$

Proposition 2.5. *Let* U *be a right antisymmetric* $A-$ *module and*

$$A \cup U \to A$$

be a pairing of $A-$ *modules:*

$$a \circ (b \cup u) = (a \circ b) \cup u, \tag{5}$$

$$(a \cup u) \circ b = (a \circ b) \cup u + a \circ (u \circ b),$$

for any $a, b \in A, u \in U$ *(about cup-products see* [6]*). Suppose that any element of* A *can be presented by a cup-product as a* $z \cup u$, *for some* $u \in U$ *and* $z \in Z(A)$. *Then for any* $a_1, \dots, a_{n-1} \in N(Z(A))$, *and* $a_n \in A$,

$$a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n \circ a = (a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n) \circ a.$$

Proof. Let $a = z \cup u$. Then by (5), and corollary 2.4,

$$a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ (a_n \circ a)) \cdots) =$$

$$a_1 \circ \{(a_2 \circ \cdots (a_{n-1} \circ ((a_n \circ z) \cup u))\} =$$

$$a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ (a_n \circ z)) \cdots) \cup u =$$

$$\{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ z) \cup u =$$

$$\{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ (z \cup u) =$$

$$\{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ a. \bullet$$

Proposition 2.6. $Z^{l.ass}(A) \subseteq N(Z(A))$.

Proof. Let $a \in Z^{l.ass}(A)$. Then for any $z \in Z(A)$,

$$(a \circ z) \circ b = a \circ (z \circ b) - (a, b, z) = 0,$$

for any $b \in A$.●

Example 1. Any associative algebra is right-symmetric. As associative algebra the matrix algebras gl_n gives us examples of right-symmetric algebras.

Example 2. Let us give associative commutative algebra U with commuting derivations $\mathcal{D} = \{\partial_i, i = 1, \dots, n\}$. Then an algebra of derivations $UD = \{u\partial_i : u \in U, \partial_i \in \mathcal{D}\}$ with multiplication $u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i$, is right-symmetric. Because of Lie algebras for $U\mathcal{D}$ are Witt algebras:

$$[u\partial_i, v\partial_j] = -u\partial_i(v)\partial_j + v\partial_j(u)\partial_i,$$

we call such algebras as right-symmetric Witt algebras.

Let Γ_n be a set of $n-$ typles $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_i are integers. Let Γ_n^+ be its subset consisting of such α, that $\alpha_i \geq 0, i = 1, \dots n$. In the case of $p = char\mathcal{K} > 0$, we consider a subset $\Gamma_n(\mathbf{m}) = \{\alpha : 0 \leq \alpha_i < p^{m_i}, i = 1, \dots, n\}$, where $\mathbf{m} = (m_1, \dots, m_n), m_i > 0, m_i \in \mathbf{Z}, i = 1, \dots, n$.

For $char\,\mathcal{K} = 0$ suppose that

$$U = \mathcal{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \{x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i} : \alpha \in \Gamma_n\}$$

is an algebra of Laurent polynomials and

$$U^+ = \mathcal{K}[x_1, \dots, x_n] = \{x^\alpha : \alpha \in \Gamma_n^+\},$$

its subalgebra of polynomials.

Let

$$O_n(\mathbf{m}) = \{x^{(\alpha)} = \prod_i x_i^{(\alpha_i)} : \alpha \in \Gamma_n(\mathbf{m}), i = 1, \dots, n\}$$

be divided power algebra if $char\,\mathcal{K} = p > 0$. Recall that $O_n(\mathbf{m})$ is p^m -dimensional and the muliplication is given by

$$x^{(\alpha)} x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha+\beta)},$$

where $m = \sum_i m_i$, and

$$\binom{\alpha+\beta}{\alpha} = \prod_i \binom{\alpha_i + \beta_i}{\alpha_i}, \; \binom{n}{l} = \frac{n!}{l!(n-l)!}, \; n, l \in \mathbf{Z}_+.$$

Let $\epsilon_i = (0, \dots, \underset{i}{1}, \dots, 0)$. Define ∂_i as a derivation of U,

$$\partial_i(x^\alpha) = \alpha_i x^{\alpha - \epsilon_i}, \; p = 0,$$

$$\partial_i(x^{(\alpha)}) = x^{(\alpha-\epsilon_i)},\ p>0.$$

Denote right-symmetric algebras $U\mathcal{D}, U^+\mathcal{D}$ for $U=\mathcal{K}[x^{\pm 1},\dots,x_n^{\pm 1}]$ as W_n^{rsym} and W_n^{+rsym}. Denote, similarly, right-symmetric algebra $O_n(\mathbf{m})\mathcal{D}$ as $W_n(\mathbf{m})^{rsym}$. Notice that Lie algebra W_n is isomorphic to a Lie algebra of formal vector fields on $n-$ dimensional torus and W_n^+ is isomorphic to a Lie algebra of formal vector fields on $\mathcal{K}^n$. As in the case of Lie algebras, $A=W_n^{rsym}$ has a grading

$$A=\oplus_k A_k,\quad A_k\circ A_l\in A_{k+l},\quad k,l\in\mathbf{Z},$$

$$A_k=\{x^\alpha\partial_i : |x^{(\alpha)}|=|\alpha|=\sum_{i=1}^n\alpha_i=k+1\}.$$

This grading induces gradings in W_n^{+rsym} and $W_n^{rsym}(\mathbf{m})$.

Example 3. Let A be an associative algebra, $C^*(A,A)=\oplus_k C^k(A,A)$, and $C^k(A,A)=\{\psi : A\times\cdots\times A\to A\}$ be a space of polylinear maps with $k-$ arguments, if $k>0$, $C^0(A,A)=A$, and $C^k(A,A)=0$, if $k<0$. Endow $C^*(A,A)$ by a "shuffle-product" multiplication:

$$C^*(A,A)\times C^*(A,A)\to C^{*-1}(A,A),$$

$$\psi\circ\phi(a_1,\dots,a_{k+l+1}) = \sum_{\substack{\sigma\in Sym_{k+l+1},\\ \sigma(1)<\cdots<\sigma(k+1),\\ \sigma(k+2)<\cdots<\sigma(k+l+1)}} \psi(\phi(a_{\sigma(1)},\dots,a_{\sigma(k+1)}),a_{\sigma(k+2)},\dots,a_{\sigma(k+l+1)}),$$

where $\psi\in C^{k+1}(A,A),\phi\in C^{l+1}(A,A),\psi\circ\phi\in C^{k+l+1}(A,A),k,l\geq 0.$

Let $\Delta_k\in C^{k+1}(A,A),k\geq 0,$ be a standard skew-symmetric polynomial:

$$\Delta_k(a_1,\dots,a_{k+1})=\sum_{\sigma\in Sym_{k+1}} sign\,\sigma\, a_{\sigma(1)}\circ\cdots\circ a_{\sigma(k+1)}.$$

Then [4]

$$\Delta_i\circ\Delta_{2k}=(i+1)\Delta_{2k+i},$$
$$\Delta_{2k+1}\circ\Delta_{2l+1}=0,$$
$$\Delta_{2k}\circ\Delta_{2l+1}=\Delta_{2k+2l+1},$$

for any $k,l,i\geq 0$.

Therefore, the algebra of standard polynomials under shuffle-product is isomorphic to the right-symmetric algebra $A_0\oplus A_1$, such that

$$A_0=\{e_i : i\geq 0\}, A_0\circ A_0\subseteq A_0,$$
$$e_i\circ e_j=(i+1/2)e_{i+j}, 0\leq i,j,$$
$$A_1=\{x^{j+1} : j\geq 0\}, A_1\circ A_1=0, A_0\circ A_1\subseteq A_1, A_1\circ A_0\subseteq A_1,$$
$$x^{i+1}\circ x^{j+1}=0, e_i\circ x^{j+1}=(1/2)x^{i+j+1},\quad i,j\geq 0.$$

The isomorphism can be given by

$$e_i \mapsto \Delta_{2i}/2, \quad x^{j+1} \mapsto \Delta_{2j+1},$$

where $i, j = 0, 1, 2, \ldots$.

This algebra has also multiplication $\cup : C^*(A,A) \otimes C^*(A,A) \to C^*(A,A)$, called as a cup-product:

$$\psi \cup \phi(a_1, \ldots, a_{k+l}) = \sum_{\substack{\sigma \in Sym_{k+l} \\ \sigma(1) < \cdots \sigma(k), \\ \sigma(k+1) < \cdots \sigma(k+l)}} \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k)})\phi(a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}).$$

Then

$$\Delta_i \cup \Delta_j = \Delta_{i+j+1},$$

for any $i, j \geq 0$. In particular, a subalgebra generated by Δ_i, is a commutative associative algebra under cup-product. These multiplications satisfy the following conditions

$$(\Delta_i \cup \Delta_j) \circ \Delta_k = (-1)^{k(j-1)}(\Delta_i \circ \Delta_k) \cup \Delta_j + \Delta_i \cup (\Delta_j \circ \Delta_k),$$

$$(a \cup b) \cup c = a \cup (b \cup c),$$

$$a \cup b = b \cup a,$$

where $b \in C^k(A,A), c \in C^j(A,A)$. So, an algebra of standard polynomials has a structure of Poissson-Novikov algebras in the sense of [5].

3. Novikov algebras

A right-symmetric algebra A is called (right) Novikov, if it satisfies the following identity

$$a \circ (b \circ c) = b \circ (a \circ c), \quad \forall a, b, c \in A.$$

Notice that algebras $W_1^{rsym}, W_1^{+rsym}, W_1^{rsym}(m)$ are Novikov. The natural question about generalisation of Novikov identities that includes the case $n > 1$ is arises. We suggest two ways to solve this problem.

In the first way, we will consider two kinds of multiplications on Witt algebras

$$u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i,$$

$$u\partial_i * v\partial_j = \partial_i(u)v\partial_j.$$

They satisfy the following identities

$$a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b = 0,$$

$$a * (b * c) - b * (a * c) = 0,$$

$$a \circ (b * c) - b * (a \circ c) = 0,$$

$$(a*b-b*a-a\circ b+b\circ a)*c=0,$$

$$(a\circ b-b\circ a)*c+a*(c\circ b)-(a*c)\circ b-b*(c\circ a)+(b*c)\circ a=0.$$

In the case of $n=1$ these multiplications coincide and all identities are reduced to two: right-symmetric identity and Novikov identity. So, an algebra A with multiplications $\circ$ and $*$ can be considered as a Novikov algebras in the general case.

The second way concerns identities of right-symmetric Witt algebras.

4. Right-symmetric identities

Let $R_n=\mathcal{K}<t_1,\dots,t_n>$ be a free algebra in the category of right-symmetric algebras generated on the set $\{t_1,\dots,t_n\}$. It can be defined in the following way. Consider a magma algebra $M<X_1,\dots,X_n>$ generated on the set $\{X_1,\dots,X_n\}$ [3]. Then R_n is a factor algebra of $M<X_1,\dots,X_n>$ over an ideal generated by associators $(X_i,X_j,X_s),\ \ i,j,s=1,\dots,n$. Elements of R_n corresponding to $X_1,\dots,X_n$ are denoted by $t_1,\dots,t_n$.

A polynomial $f(t_1,\dots,t_n)\in\mathcal{K}<t_1,\dots,t_n>$ is called *a right-symmetric identity* on A, if $f(a_1,\dots,a_n)=0$, for any $a_1,\dots,a_n\in A$. If $f(a_1,\dots,a_n)=0$, is an identity on A, then $f(a_{\sigma(1)},\dots,a_{\sigma(n)})=0$ is also an identity on A, for any permutation $\sigma\in Sym_k$.

Further, denotions like $x_1\circ x_2\circ\cdots\circ x_{k-1}\circ x_k$ will mean an element $x_1\circ(x_2\circ(\cdots\circ(x_{k-1}\circ x_k)\cdots))$. For any finite sequence of integers $\mathbf{i}=(i_1,i_2,\dots,i_k)$, with $i_l=1,\dots,n$, and $l=1,\dots,k$, where k is any integer, set $t^{\mathbf{i}}=t_{i_1}\circ\dots t_{i_k}$. Elements of the form $\lambda_{\mathbf{i}}t^{\mathbf{i}}$, where $\lambda_{\mathbf{i}}\in\mathcal{K}$, are called as (left) monoms. If $\lambda_{\mathbf{i}}=0$, then it is a trivial monom. If we would like to pay attention to monoms with $\lambda_{\mathbf{i}}\neq 0$, then we call $\lambda_{\mathbf{i}}t^{\mathbf{i}}$ as a nontrivial monom. The sum of monoms is called as a (left) polynomial. A space of (left) polynomials is denoted by R_n^{left}. In the universal enveloping algebra $U(R_n)$ they correspond to polynomials generated by $l_{t_i},t_i\in R_n$. *A degree* of the nontrivial monom $\lambda_{\mathbf{i}}t_{i_1}\circ\cdots\circ t_{i_k}$, by definition is k :

$$deg\, t^{\mathbf{i}}=|\mathbf{i}|=\sum_l i_l.$$

For a polynomial

$$f=\sum_{\mathbf{i}}\lambda_{\mathbf{i}}t^{\mathbf{i}}\in R_n^{left},$$

we will say that f has a monom $\lambda_{\mathbf{i}}t^{\mathbf{i}}$, if $\lambda_{\mathbf{i}}\neq 0$. The maximal degree of the nontrivial monoms of f is called as a *degree* of the polynomial f :

$$def\, f=max\{|\mathbf{i}|:\lambda_{\mathbf{i}}\neq 0\}.$$

Suppose that A is a graded right-symmetric algebra:

$$A = \oplus_i A_i, \quad A_i \circ A_j \subseteq A_{i+j}.$$

Elements of A_i are called as homogeneous elements of the weight i. Notation $|a| = i$, will means that a is a homogeneous element of A and $a \in A_i$. The element obtained from $t^{\mathbf{i}}$ by substituting $t_i := a_i$, we denote as $a^{\mathbf{i}}$. For graded right symmetric algebra A and for any monom $t^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \ldots, i_k)$, it is evident that

$$a^{\mathbf{i}} \in A_{\sum |a_{i_1}| + \cdots + |a_{i_k}|}, \tag{6}$$

In our paper we deal only in left polynomial identities.

The following left polynomial of R_{k+1}^{left} is called a *right-symmetric standard polynomial* of degree $k+1$:

$$s_k^{rsym}(t_1, \ldots, t_{k+1}) = \sum_{\sigma \in Sym_k} sign\, \sigma t_{\sigma(1)} \circ \ldots \circ t_{\sigma(k)} \circ t_{k+1}.$$

Notice that it can be considered as an associative standard polynomial of degree k for left multiplications in the universal enveloping algebra $U(R_{k+1})$ [6]. If

$$s_k^{ass}(l_{t_1}, \ldots, l_{t_k}) = \sum_{\sigma \in Sym_k} sign\, \sigma\, l_{t_{\sigma(1)}} \ldots l_{t_{\sigma(k)}},$$

then

$$s_k^{rsym}(t_1, \ldots, t_k, t_{k+1}) = (t_{k+1}) s_k^{ass}(l_{t_1}, \ldots, l_{t_k}).$$

If $f(a_1, \ldots, a_k) = 0$ is a right-symmetric polynomial identity of A, then the polynomial g obtained from f by permutation of arguments $g(t_1, \ldots, t_k) = f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}), \quad \sigma \in Sym_k$, give also a right-symmetric polynomial identity. Denote g as σf. Let $\tau_l \in Sym_{2n+1}$, $l = 0, 1, \ldots, 2n$, be the cyclic permutation of elements $0, 1, \ldots, 2n$: $\tau_l = (l, 0, 1, \ldots, l-1)$.

The following is our main result.

Theorem 4.1. *Let A be one of the following right-symmetric algebras $W_n^{rsym}(p=0), W_n^{+rsym}(p=0), W_n^{rsym}(\mathbf{m})(p>0)$.*

i) A satisfies the right-symmetric standard identity of degree $2n+1$:

$$\sum_{\sigma \in Sym2n} sign\, \sigma\, a_{\sigma(1)} \circ a_{\sigma(2)} \circ \cdots a_{\sigma(2n)} \circ a_{2n+1} = 0,$$

for any $a_1, \ldots, a_{2n+1} \in A$. In particular, the polynoms $\tau_l s_{2n}^{rsym}, l = 0, 1, \ldots, 2n$, are also right-symmetric identities of A.

ii) If $char\, \mathcal{K} = 0$, then the algebra $A = W_n^{rsym}$, or W_n^{+rsym}, has no left polynomial identity of degree less that $2n+1$. If $p > 0$, then the algebra $W_n(\mathbf{m})$ has no multilinear polynomial identity of degree less than $2n+1$.

iii) Let f be any left polynomial identity of A of degree $2n+1$. In the case of $W_n(\mathbf{m}), p>0$, we suppose that f is also multilinear. Then f is a linear combination of $\tau_l s_{2n}^{rsym}$, $l=0,1,\dots,2n$.

5. Identities of right-symmetric Witt algebras.

In this section $A = W_n^{rsym}, W_n^{+rsym}$, or $W_n^{rsym}(\mathbf{m}), p>0$. Define $s_{k,r}\in C^k(A,A), r=1,\dots,n$, by

$$s_{k,r}(u_1\partial_{i_1},\dots,u_k\partial_{i_k}) = \sum_{\sigma\in Sym_k} sign(\sigma)\partial_r(u_{\sigma(1)})\partial_{i_{\sigma(1)}}(u_{\sigma(2)})\dots\partial_{i_{\sigma(k-1)}}(u_{\sigma(k)})\partial_{i_{\sigma(k)}}.$$

Let

$$A^+_{-1} = \{\partial_i : i=1,\dots,n\},$$
$$A^+_0 = \{x_i\partial_j : i,j,=1,\dots,n\}.$$

Lemma 5.1. [6]

$$Z(A) = A^+_{-1},$$
$$N(Z(A)) = A^+_{-1}\oplus A^+_0.$$

In particular, $N(Z(A))$ has a subalgebra A^+_0 isomorphic to Mat_n. It is an associative subalgebra of A.

The proof of theorem 4.1 is based on the following observation.

Theorem 5.2. *For any $r=1,\dots,n$, and $a_1,\dots,a_{2n}\in A$,*

$$s_{2n,r}(a_1,\dots,a_{2n}) = 0.\bullet$$

Let $U=\mathcal{K}[x_1^{\pm1},\dots,x_n^{\pm1}], \mathcal{K}[x_1,\dots,x_n]$, or $O_n(\mathbf{m})$, if $A=W_n, W_n^+$, or $W_n(\mathbf{m})$. It is enough to prove the theorem for a basic elements a_i. Recall that they have the form $u\partial_i$, where $u=x^\alpha, \alpha\in\Gamma_n$. For the proof of our theorem we need some lemmas.

Lemma 5.3. *If one of the basic elements $a_1,\dots,a_k$ belongs to A^+_{-1}, then*

$$s_{k,r}(a_1,\dots,a_k)=0.$$

Proof. Evident.

Lemma 5.4. *For any $u,v\in U$, and $1\le l\le k$,*

$$s_{k,r}(a_1,\dots,a_{l-1},uv\partial_{i_l},a_{l+1},\dots,a_k) =$$
$$v s_{k,r}(a_1,\dots,a_{l-1},u\partial_{i_l},a_{l+1},\dots,a_k)+u s_{k,r}(a_1,\dots,a_{l-1},v\partial_{i_l},a_{l+1},\dots,a_k).$$

Proof. Evident.

Lemma 5.5. *If $s_{k,r}(a_1,\dots,a_k)=0$, for all $a_1,\dots,a_k\in W_n^{+rsym}$, then $s_{k,r}(a_1,\dots,a_k)=0$, for all $a_1,\dots,a_k\in W_n^{rsym}$.*

Proof. Let l be a number of elements of W_n in the set $\{a_1, \dots, a_k\}$. We use induction reasoning on $l = n, \dots, 0$.

By the assumption of our lemma the base of induction is true. Suppose that the statement is true for l and $l-1$ elements of the arguments set $a_1, \dots, a_k$ belongs to W_n^{+rsym}. Since the polynom $s_{k,r}$ is skew-symmetric, we can assume that $a_1, \dots, a_{l-1} \in W_n^{+rsym}$, and a_l has a form $uv\partial_i$, for some $u = x^{-\alpha}, v = x^{\beta}, \alpha, \beta \in \Gamma_n^+$.

Present 1 as uu^{-1}. According lemma 9 and lemma 10

$$0 = s_{k,r}(a_1, \dots, a_{l-1}, \partial_{i_l}, \dots, a_k) =$$

$$us_{k,r}(a_1, \dots, a_{l-1}, u^{-1}\partial_{i_l}, \dots, a_k) + u^{-1}s_{k,r}(a_1, \dots, a_{l-1}, u\partial_{i_l}, \dots, a_k).$$

Since $u^{-1} = x^{\alpha} \in U^+$, according the inductive hypothesis, $s_{k,r}(a_1, \dots, a_{l-1}, u^{-1}\partial_{i_l}, \dots, a_k) = 0$. Therefore,

$$u^{-1}s_{k,r}(a_1, \dots, a_{l-1}, u\partial_{i_l}, \dots, a_k) = 0,$$

and

$$s_{k,r}(a_1, \dots, a_{l-1}, u\partial_{i_l}, \dots, a_k) = uu^{-1}s_{k,r}(a_1, \dots, a_{l-1}, u\partial_{i_l}, \dots, a_k) = 0.$$

Using once again lemma 10 and the inductive hypothesis we have

$$s_{k,r}(a_1, \dots, a_{l-1}, uv\partial_{i_l}, \dots, a_k) =$$

$$us_{k,r}(a_1, \dots, a_{l-1}, v\partial_{i_l}, \dots, a_k) + vs_{k,r}(a_1, \dots, a_{l-1}, u\partial_{i_l}, \dots, a_k) = 0.$$

So, the induction transfer is possible and the statement is proved.

Lemma 5.6. *If* $f \in R_k^{left}$, *satisfy the condition*

$$f(a_1, \dots, a_k) = 0,$$

for any $a_1, \dots, a_k \in A_{-1}^+ \oplus A_0^+$, *then*

$$f(a_1, \dots, a_k) = 0,$$

for any $a_1, \dots, a_k \in A$.

Proof. By lemma 11 it is enough to consider the cases of $A = W_n^+$, and $A = W_n(\mathbf{m})$. Our arguments in both of these cases are similar and for definitely we assume that $A = W_n^+$.

If $|a_l| < 0$, for some $1 \le l \le 2n$, by the condition of our lemma the statement is true. Suppose that $|a_l| \ge 0$, for all $l = 1, \dots, n$. We will argue by induction on $q = |a_1| + \cdots + |a_{2n}|$.

For $q = 0$, we have $|a_1| = \cdots |a_{2n}| = 0$. According to the condition of our lemma the statement is true.

Suppose that for $q - 1 \ge 0$, the statement is true and $a_1, \dots, a_{2n} \in W_n^+$, such that $|a_1| + \cdots + |a_{2n}| = q$, $|a_1| \ge 0, \dots, |a_{2n}| \ge 0$. We have

$$\partial_t(s_{2n,r}(a_1, \dots, a_{2n})) = \sum_{l=1}^{2n} s_{k,r}(a_1, \dots, a_{l-1}, \partial_t(u_l)\partial_{i_l}, \dots, a_{2n}),$$

and

$$s_{k,r}(a_1,\dots,a_{l-1},\partial_t(u_l)\partial_{i_l},\dots,a_{2n})=0,$$

since

$$|a_1|+\cdots+|a_{l-1}+|\partial_t(u_l)\partial_{i_l}|+\cdots+|a_{2n}|=q-1,\quad |\partial_t(u_l)\partial_{i_l}|\geq 0,$$

or

$$|\partial_t(u_l)\partial_{i_l}|=-1.$$

So, for any $1\leq t\leq n$,

$$\partial_t(s_{2n,r}(a_1,\dots,a_{l-1},\partial_t(u_l)\partial_{i_l},\dots,a_{2n}))=0. \tag{7}$$

Because of

$$\partial_t(w)=0, w\in U, i=1,\dots,n\Rightarrow w=\lambda 1,\ \text{ for some } \lambda\in\mathcal{K},$$

and

$$s_{2n,r}(a_1,\dots,a_{2n})\in A_q,\ q>0,$$

from (7) we have,

$$s_{2n,r}(a_1,\dots,a_{2n})=0.$$

Therefore, the induction transfer is possible and the theorem is proved completely.

Proof of theorem 5.2. If one element of the set $\{a_1,\dots,a_{2n}\}$ belongs to A^+_{-1}, then by lemma 9,

$$s_{2n,r}(a_1,\dots,a_{2n})=0.$$

Suppose now that $a_1,\dots,a_{2n}\in A^+_0=\{x_i\partial_j : i,j=1,\dots,n\}\cong Mat_n$. Let $a_l=x_{j_l}\partial_{i_l}$, $l=1,\dots,2n$. According to the Amitsur-Levitzki theorem for any $r=1,\dots,n$,

$$s_{2n,r}(a_1,\dots,a_{2n})=$$

$$\partial_r\{\sum_{\sigma\in Sym2n} sign\,\sigma\, x_{j_{\sigma(1)}}\partial_{i_{\sigma(1)}}(x_{j_{\sigma(2)}})\cdots\partial_{i_{\sigma(2n-1)}}(x_{j_{\sigma(2n)}})\partial_{i_{\sigma(2n)}}\}=0.$$

By lemma 12, theorem 5.2 is established.

Proof of theorem 4.1.

i) Let $a_l=u_l\partial_{i_l}$, $l=1,2,\dots,2n$, $a_0=u\partial_r$. Then by theorem 5.2

$$\sum_{\sigma\in Sym_{2n}} sign\,\sigma\, a_{\sigma(2n)}\circ a_{\sigma(2)}\circ\cdots a_{\sigma(1)}\circ a_0=$$

$$\sum_{\sigma\in Sym_{2n}} u\partial_r(u_1)\partial_{i_1}(u_2)\cdots\partial_{2n-1}(u_{2n})\partial_{2n}=$$

$$us_{2n,r}(a_1,\dots,a_{2n})=0.$$

ii) If $p=0$, and A has nontrivial left polynomial identity of degree d, then it has multilinear nontrivial left polynomial identity of degree $\leq d$. The proof of this statement is based on the linearisation method and it does not depend on the condition of associativity of A. Let

$f(x_1,\dots,x_d)$ be a multilinear identity of A of degree $\le 2n$. Then $f(x_1,\dots,x_n)$ has a monom of the form $\lambda x_d x_{d-1}\dots x_1$, $\lambda\ne 0$. As in the case of matrices, if $d<2n+1$, we can take $a_1=\partial_1, a_2=x_1\partial_1, a_3=x_1\partial_2, a_4=x_2\partial_2,\dots,a_d=x_m\partial_m$, if $d=2m$, and $a_d=x_{m-1}\partial_m$, if $d=2m-1$. Then

$$a_d\circ a_{d-1}\circ\cdots\circ a_1=\partial_m,$$

and for any permutation $\sigma\in Sym_d$, $\sigma\ne id$,

$$a_{\sigma(d)}\circ\ldots\circ a_{\sigma(1)}=0.$$

Therefore,

$$f(a_1,\dots,a_d)=\lambda\partial_m\ne 0.$$

Contradiction.

iii) Let f be a left polynomial of degree $2n+1$ and f depends on valuables $t_1,\dots,t_s$, i.e. $f\in R_s^{left}$. Suppose that $f=0$ is an identity on A. By lemma 12 it is equivalent to say that $f=0$ is an identity on $N(Z(A))=A_{-1}^+\oplus A_0^+$.

Let

$$f(t_1,\dots,t_s)=\sum_k\sum_{i_1,\dots,i_k=1,\dots,s}\lambda_{i_1,\dots,i_{k-1},i_k}t_{i_1}\circ\cdots t_{i_{k-1}}\circ t_{i_k}. \tag{8}$$

We would like to prove that $s=2n+1$. If f is multilinear, it is evident. Since in the case of $A=W_n(\mathbf{m})^{rsym}$, we suppose that f is multilinear, below we take $A=W_n^{rsym},W_n^{+rsym},p=0$.

Let us prove that $s\ne 1$. Suppose that it is false. Let

$$\mu_k=\lambda_{\underbrace{1,\dots,1}_{k\ \text{times}}},\ \text{and}\ t_1^k=\underbrace{t_1\circ\cdots\circ t_1}_{k\ \text{times}}.$$

Then

$$f(t_1)=\sum_k\mu_k t_1^k.$$

Substitute $a_1=x_1^2\partial_1$ instead of t_1. Since

$$a_1^{\circ k}=2^{k-1}x^{k+1}\partial_1,$$

we have

$$\sum_k\mu_k 2^{k-1}x_1^{k+1}\partial_1=0,$$

or

$$\mu_k2^{k-1}=0,$$

for any k. Therefore, $\mu_k=0$, for any k, and f is a trivial polynomial, i.e., $f=0$. So, the case $s=1$ is not possible.

Suppose that all left polynomials of degree $2n+1$ with $s-1$ valuables are trivial. Define $f_s \in R_s^{left} \subseteq M < t_1, \dots, t_s >$ by

$$f_s(t_1, \dots, t_s) = \sum_k \sum_{i_1,\dots,i_{k-1}=1,\dots,s-1} \lambda_{i_1,\dots,i_{k-1},s} t_{i_1} \circ \cdots t_{i_{k-1}} \circ t_s.$$

Denote by $\bar{f}_s$ the sum of monoms $\lambda_{\mathbf{i}} t^{\mathbf{i}}$, such that all coordinates of $\mathbf{i} = (i_1, \dots, i_k)$ differ from s. So, f_s has monoms $\lambda_{\mathbf{i}} t^{\mathbf{i}}$, such that the entrance of t_s is exactly 1 and t_s appears only at the end of monoms. As far as $\bar{f}_l \in R_s^{left}$, its monoms do not depend from t_s. So $\bar{f}_l \in R_{s-1}^{left}$.

It is evident that $a_{i_1} \circ \cdots \circ a_{i_k} = 0$, if $a_{i_l} \in Z(A)$, for some $l < k$. Thus for any $z \in Z(A)$, we have

$$0 = f(a_1, \dots, a_{s-1}, z) = f_s(a_1, \dots, a_{s-1}, z) + \bar{f}_s(a_1, \dots, a_{s-1}). \tag{9}$$

By corollary 2.4, for any $a_1, \dots, a_{s-1} \in N(Z(A))$,

$$f_s(a_1, \dots, a_{s-1}, z) = F_s(a_1, \dots, a_{s-1}) \circ z,$$

for some $F_s \in R_{s-1}^{left}$. Namely,

$$F_s(x_1, \dots, x_{s-1}) = \sum_k \sum_{i_1,\dots,i_{k-1}=1,\dots,s-1} \lambda_{i_1,\dots,i_{k-1},s} t_{i_1} \circ \cdots t_{i_{k-1}},$$

if f_s has a form (8).

Thus, the condition (9) can be written in the following way

$$F_s(a_1, \dots, a_{s-1}) \circ z + \bar{f}_l(a_1, \dots, a_{s-1}) = 0. \tag{10}$$

The first summand of the left hand of (10) depends on z linearly and the second summand does not depend on z. Therefore,

$$F_s(a_1, \dots, a_{s-1}) \circ z = 0, \tag{11}$$

$$\bar{f}_s(a_1, \dots, a_{s-1}) = 0, \tag{12}$$

for any $z \in Z(A)$ and $a_1, \dots, a_{s-1} \in N(Z(A))$.

By (12) and lemma 12 the polynomial $\bar{f}_s$ gives us an identity of degree no more that $2n+1$. If this degree is less than $2n+1$, according (ii), $\bar{f}_s$ should be a trivial polynomial. If $deg\, \bar{f}_s = 2n+1$, then by the inductive suggestion $\bar{f}_s$, as a polynomial with $s-1$ valuables, also should be trivial.

Notice that $deg\, f_s \le 2n+1$, and $def\, F_s \le 2n$. Since the centraliser of $Z(A)$ coincides with $Z(A)$ (see lemma 5.1) and $Z(A) \subset A_{-1}$, by (6) from (11) we obtain that

$$F_s(a_1, \dots, a_{s-1}) = 0$$

is an identity for $N(Z(A))$. In particular, it is an identity on $A_0^+ \subset N(Z(A))$ (recall that $N(Z(A)) = A_{-1}^+ \oplus A_0^+$). We have mentioned

that $A_0^+ = \{x_i\partial_j : i,j = 1,\dots,n\} \cong Mat_n$. So, $F_s = 0$ is an identity on Mat_n of a degree no more than $2n$.

By the Amitsur-Levitzki theorem, the minimal degree of nontrivial polynomial identity on Mat_n is $2n$ and any such polynomial should be a standard polynomial of degree $2n$ up to scalar. Therefore, $s-1 = 2n$. Moreover,

$$F_{2n+1}(a_1,\dots,a_{2n}) = \gamma_{2n+1} s_{2n}^{ass}(a_1,\dots,a_{2n}),$$

where

$$s_{2n}^{ass}(t_1,\dots,t_{2n}) = \sum_{\sigma\in Sym_{2n}} sign\,\sigma\, t_{\sigma(1)}\circ\cdots\circ t_{\sigma(2n)}),$$

and $\gamma_{2n+1}\in\mathcal{K}$.

So, for

$$g_{2n+1}(a_1,\dots,a_{2n},a_{2n+1}) =$$
$$f_{2n+1}(a_1,\dots,a_{2n},a_{2n+1}) - \gamma_{2n+1}\, s_{2n}^{rsym}(a_1,\dots,a_{2n},a_{2n+1})$$

and for any $a_{2n+1} = u\partial_r \in A$, $a_1,\dots,a_{2n}\in A_{-1}^+\oplus A_0^+$, by proposition 2.5 we have

$$g_{2n+1}(a_1,\dots,a_{2n},a_{2n+1}) =$$
$$f_{2n+1}(a_1,\dots,a_{2n},a_{2n+1}) - \mu\, s_{2n}^{rsym}(a_1,\dots,a_{2n},a_{2n+1}) =$$
$$u\partial_r\{F_{2n+1}(a_1,\dots,a_{2n})\} - \gamma_{2n+1}\, u\partial_r\{s_{2n}^{ass}(a_1,\dots,a_{2n})\} =$$
$$u\partial_r\{F_{2n+1}(a_1,\dots,a_{2n}) - s_{2n}^{ass}(a_1,\dots,a_{2n})\} =$$
$$0.$$

By lemma 12,

$$g_{2n+1}(a_1,\dots,a_{2n+1}) = 0,$$

for any $a_1,\dots,a_{2n+1}\in A$. Thus,

$$f_s = \gamma_s s_{2n}^{rsym},$$

for $s = 2n+1$.

For any $l = 1,\dots,2n$, let f_l be a sum of monoms $\lambda_{\mathbf{i}} t^{\mathbf{i}}$ of f, such that $\mathbf{i} = (i_1,\dots,i_k)$ ends by l and $i_1,\dots,l_{k-1}\neq l$:

$$f_l(t_1,\dots,t_{2n+1}) = \sum_k \sum_{i_1,\dots,i_{k-1}=1,\dots\hat{l},\dots,2n+1} \lambda_{i_1,\dots,i_{k-1},l} t_{i_1}\circ\cdots t_{i_{k-1}}\circ t_l.$$

Recall that the denotion $\hat{x}$ means that x is omitted. Let $\bar{f}_l$ be a sum of monoms $\lambda_{\mathbf{i}} t^{\mathbf{i}}$, such that all coordinates of $\mathbf{i} = (i_1,\dots,i_k)$ differ from l. Reorder valuables $t_1,\dots,t_l,\dots,t_{2k+1}$, in a such way, that the last valuable will be t_l. Repeating the arguments given above we can see that polynomial $\bar{f}_l$ is trivial and f_l is equal to a polynomial obtained from $\gamma_l s_{2n}^{rsym}$ by permutation of arguments, where $\gamma_l\in\mathcal{K}$.

Therefore, f_{2n+1} is a linear combination of standard polynomials $\tau_l s_{2n}^{rsym}, l = 0,1,\dots,2n$. $\bullet$

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